Note regarding water storage gradient for review of doi:10.5194/bg-2022-218

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In the manuscript, the groundwater flow equation in two dimensions is given as

$$\frac{\partial \Theta}{\partial t} = \nabla\left(D\nabla\Theta\right) + P - ET \tag{5}$$

(equation numbers from text). In Appendix B, equation (5) is said to follow from the groundwater flow equation in terms of hydraulic head h

$$S_y\frac{\partial h}{\partial t} = \nabla\left(T\nabla h\right) + P - ET \tag{B1}$$

based on a change of variables via

$$S_y = \frac{\partial \Theta}{\partial h}. \tag{B2}$$

However, the statements (5) and (B1) are not equivalent because it is not generally true that the horizontal flow $T\nabla h$ can be written $D\nabla\Theta$ with $D = T/S_y$, and with S_y a function of the water table depth, if there are gradients in the surface elevation. An extra term arises from the gradient in surface elevation p combined with the non-uniform profile of specific yield S_y, so that $\nabla\Theta \neq S_y\nabla h$.

The specific yield is defined in the text as

$$S_y(\zeta) = \begin{cases} 1 & \zeta > 0 \\ s_1 e^{s_2\zeta} & \zeta \leq 0, \end{cases} \tag{8}$$

and the water storage Θ is given in Appendix B by

$$\Theta(\zeta) = \begin{cases} \zeta + \frac{s_1}{s_2}\left(1 - e^{-s_2 p}\right) & \zeta > 0 \\ \frac{s_1}{s_2}\left(e^{s_2\zeta} - e^{-s_2 p}\right) & \zeta \leq 0 \end{cases}$$

(equation B5 in the text, with substitution from equation B6 and the inline equation in the following line), equivalent to

$$\Theta(\zeta) = \int_{\zeta'=-p}^{\zeta} S_y\, d\zeta'.$$

Here is the relevant rule from calculus:

$$\nabla\int_{a(\mathbf{x})}^{b(\mathbf{x})} f(\zeta')\, d\zeta' = f(b)\nabla b - f(a)\nabla a$$

Therefore, the gradient in water storage is computed as

$$\nabla\Theta = \nabla\int_{\zeta'=-p}^{\zeta} S_y(\zeta')\, d\zeta' = S_y(\zeta)\nabla\zeta + S_y(-p)\nabla p$$

Rearranging,

$$\nabla\Theta = S_y(\zeta)\nabla h + [S_y(-p) - S_y(\zeta)]\,\nabla p.$$

Thus, a term $\left[\frac{S_y(-p)}{S_y(\zeta)} - 1\right] T\nabla p$ is effectively added to the horizontal flow term $T\nabla h$ when equation 5 is used instead of equation B1. This term may be big or small depending on the ratio of the specific yield at the water table and at $z = 0$, and the relative magnitude of the gradients in the water table h and peat surface p.